FH
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KC
KA

Name: ____________________

Class: 12MT2__ or 12MTX__

Teacher: ____________________

**CHERRYBROOK TECHNOLOGY HIGH SCHOOL**

**2011 AP4**

**YEAR 12 TRIAL HSC EXAMINATION**

# MATHEMATICS

*Time allowed* - 3 HOURS
*(Plus 5 minutes reading time)*

## DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page and must show your name and class.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Standard Integral Tables are provided.
- Write your name and class in the space provided at the top of this question paper.
- Your solutions will be collected in one bundle stapled in the top left corner. Please arrange them in order, Q1 to 10. The exam paper must be handed in with your solutions.

**QUESTION 1** (12 marks) **MARKS**

a) Factorise fully $x^3 - 4x$ 1

b) Find the centre and radius of the circle 2

$x^2 + y^2 - 4y = 0$

c) Solve $|4x + 1| = 5$ 2

d) Find the values of a and b such that $\left(\sqrt{a} + \sqrt{2}\right)^2 = 5 + 2\sqrt{b}$ 2

e) Let $f(x) = \sqrt{4 - x^2}$. What is the domain of f(x)? 1

f) Find the exact values of θ such that $\sqrt{2}\,\sin\theta = 1$, where $0 \le \theta \le 2\pi$ 2

g) Express $\frac{3}{2a+3} - \frac{a-4}{a}$ as a single fraction in simplest form. 2

**QUESTION 2** (12 marks) Start a new page **MARKS**

a) Differentiate with respect to x:

i) $y = \frac{\ln x}{x^2}$ 2

ii) $f(x) = (1 + \sin x)^6$ 2

b) i) Find $\int (2x+1)^3 dx$ 1

ii) Find $\int \frac{6}{3x+1} dx$ 2

iii) Given that $\int_0^2 (px+1)\,dx = 1$, where p is a constant, 2

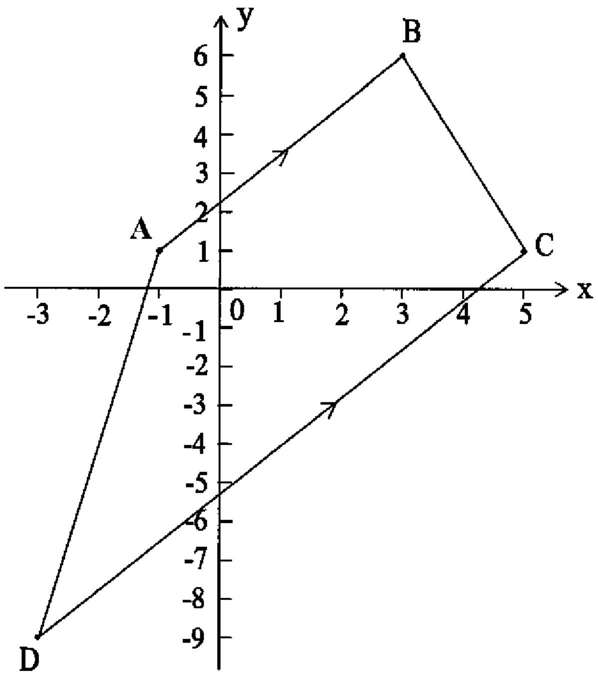
find the value of p.

c) Find the values of $k$ for which the equation $kx^2 - (k+3)x + 4 = 0$ has two distinct real roots. 3

**QUESTION 3** (12 marks) Start a new page **MARKS**

a) The area under the curve $y=\sqrt{16-x^2}$, from $x = 0$ to $x=3$, is rotated about the $x$-axis. Find the volume of the solid of revolution formed. 2

b) The diagram shows the points A (-1, 1), B(3, 6), C (5, 1) and D (-3, -9). AB is parallel to DC.




i) Find the coordinates of E, the midpoint of DC. 1

ii) Find the equation of BE. 2

iii) Find the perpendicular distance from A to the line BE. 1

iv) Show that ABED is a parallelogram and find its area. 3

c) Find the value of $x$ if $\frac{9}{x}+\frac{9}{x^2}+\frac{9}{x^3}+\ldots\ldots\ldots=18$ 3

**QUESTION 4** (12 marks) Start a new page **MARKS**

a) Anthony is a weight lifter. He is training for a competition in 4 weeks. On the first day he lifts 120kg and each day after that he lifts 1.5kg more than the previous day, until the day he reaches his goal of lifting 150kg. He then continues to lift 150kg each day.

i) How much does Anthony lift on the $10^{th}$ day? 1

ii) On which day does he first lift 150kg? 1

iii) Anthony states that the total weight he lifted over these 28 days is more than the weight of an elephant of 3.5 tonnes. Is he correct? Give reasons to your answer. 2

b) Use Simpson's Rule, with 5 function values, to find an approximation to 3

$$\int_0^4 x e^x \, dx$$

c) Find the size of each interior angle of a regular polygon with 20 sides. 2

d) For the equation $x^2 - 8y + 2x + 9 = 0$, find:

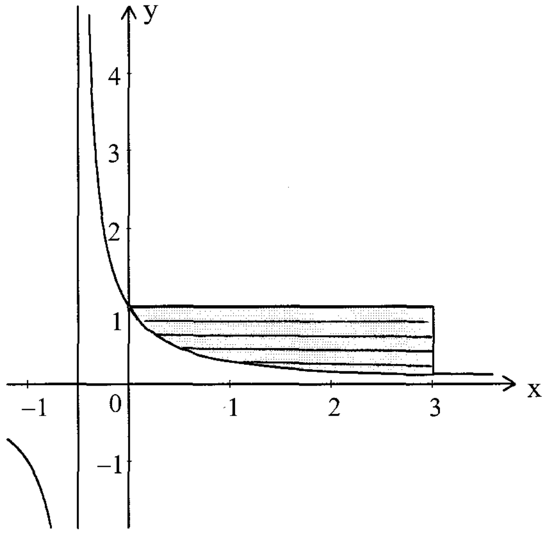
i) the vertex 2

ii) the equation of the directrix. 1

**QUESTION 5** (12 marks) Start a new page **MARKS**

a) The diagram shows the area bounded by $y = \dfrac{1}{2x+1}$, the line $x = 3$ and the line $y = 1$. 3

Find the shaded area.

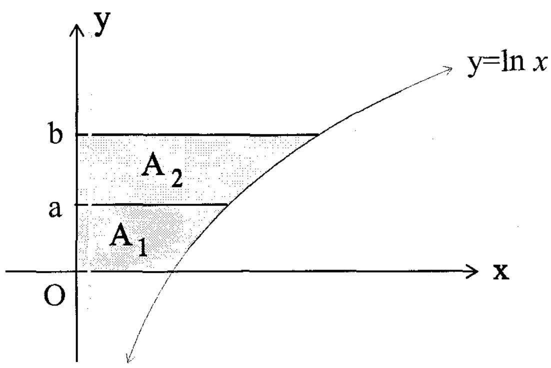



b) Find the value of x if $\log_{10}(2x+4) = 1 + \log_{10} x$ **2**

c) Find the equation of the tangent to the curve $y = x + e^{2x}$ at the point where $x = 0$. 3

d) In the diagram the shaded areas $A_1$ and $A_2$ are bounded by $y = \ln x$ and the y axis. **4**
But $A_1$ is also bounded by $y = 0$ and $y = a$ and $A_2$ is also bounded by $y = a$ and $y = b$.

$A_1$ is 1 square unit and $A_2$ is 2 square units.

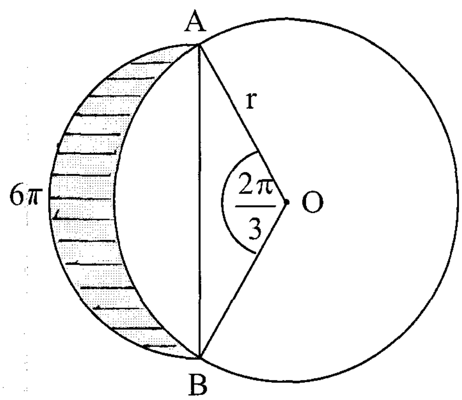



Find the values of $a$ and $b$.

**QUESTION 6** (12 marks) Start a new page **MARKS**

a) In the diagram, AB is a chord of a circle with centre O and radius r cm, such that $\angle AOB = \frac{2\pi}{3}$.

AB is also the diameter of a semicircle with arc length $6\pi$ cm.




i) Find the length of the interval AB. **1**

ii) Find r the radius of the circle centred at O. **1**

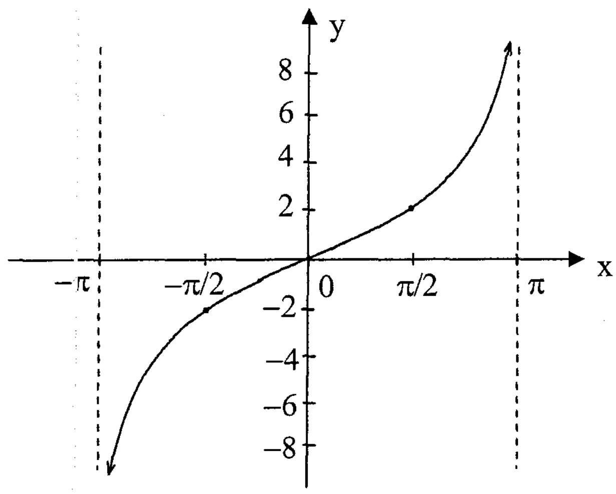
iii) Find the shaded area, which lies inside the semicircle, but outside the circle. **3**

b) i) Prove that $\dfrac{\sin^2\theta}{1-\sin^2\theta} = \sec^2\theta - 1$ **2**

ii) Hence, or otherwise, find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sin^2\theta}{1-sin^2\theta}\,d\theta$. **2**

**QUESTION6 Continued** **Marks**

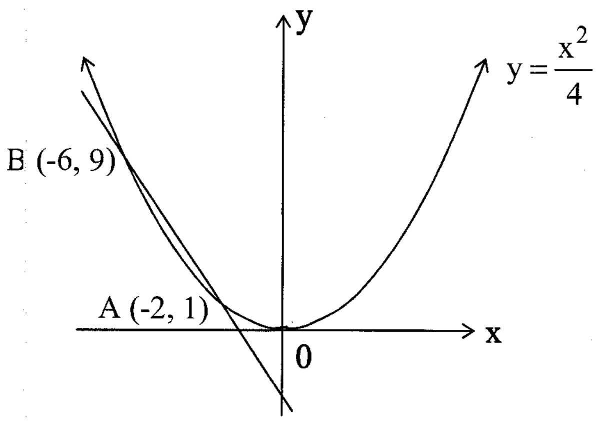
c) The diagram shows the graph of $y = a \tan bx$, for $-\pi < x < \pi$. Find the values of a and b. 2




d) Graph $y = 2\cos 2x$, for $0 \le x \le 2\pi$. 1

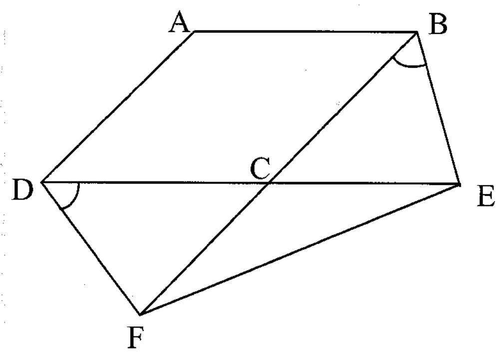
**QUESTION 7** (12 marks) Start a new page **MARKS**

a) A(-2, 1) and B(-6, 9) are two points on the parabola $y = \frac{x^2}{4}$.




Find the coordinates of the point C on the parabola, where the normal is parallel to the line AB. 2

b) In the diagram ABCD is a rhombus. DC is produced to E and BC is produced to F, such that $\angle CDF = \angle CBE$. 3




Prove that $\Delta$ CEF is isosceles.

c) A function f(x) is defined as $f(x) = x^3 + 6x^2 + 15x$ for $-3 \leq x \leq 1$.

i) Show that the curve of $y = f(x)$ is always increasing. **2**

ii) Find the coordinates of the point of inflexion. **2**

iii) Sketch the curve $y = f(x)$, clearly indicating the intercepts and the point of inflexion. **2**

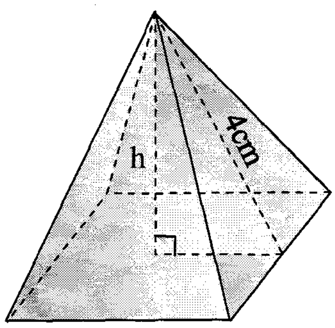
iv) Find the range of f(x). **1**

**QUESTION 8** (12 marks) Start a new page **MARKS**

a) Let $f(x) = x^4 + px^3 - 6x^2 - 2$, where p is a constant. 2

Find the values of p for which the graph of $y = f(x)$ is concave up at $x = 2$.

b) A diamond is to be cut in the shape of a square pyramid, with a slant height 4 cm and a perpendicular height of $h$ as shown in the diagram.




i) Show that the volume of the diamond can be expressed as **2**

$V = \frac{4h}{3}\left(16 - h^2\right)$.

ii) Find the greatest volume of such a diamond. **2**

c) Air pressure $P$, measured in kPa, at an altitude of $h$ metres above sea level can be approximated using the formula $P = 101\, e^{-kh}$, where k is a constant. The air pressure is 90 kPa at an altitude of 1000 m.

i) Show that the formula satisfies the equation $\frac{dP}{dh} = -kP$. 1

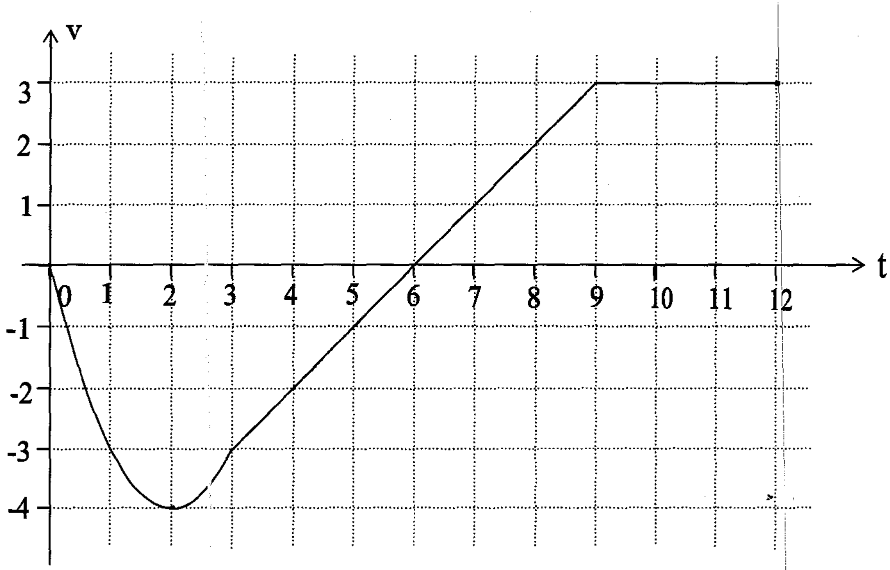
ii) Find the air pressure at an altitude of 5000m. 3

iii) Find the depth of a mine, below sea level, where the air pressure is 103 kPa. 2

**QUESTION 9** (12 marks) Start a new page **MARKS**

a) A particle moves along the x axis. Initially, it is at rest at the origin.

The graph shows its velocity, in $ms^{-1}$, as a function of time t for $0 \le t \le 12$.




i) Given that the velocity $v = t^2 - 4t$ for $0 \le t \le 3$, find the distance travelled by the particle during this time. 2

ii) Show that the particle returns to the origin after 12 seconds. 2

iii) Sketch the graph of the displacement of this particle for $0 \le t \le 12$. 2

b) The velocity of a particle is given by $v = 2e^t - 6e^{-t} - 1$, where v is the velocity in $ms^{-1}$ and t is the time in seconds.

The particle is initially at a point 9m to the right of the origin.

i) Find the initial velocity 1

ii) Find the time when the particle comes to rest. 2

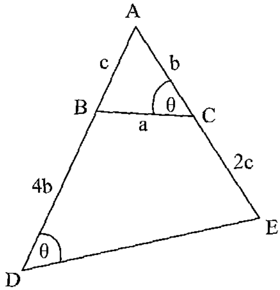
iii) Find the acceleration of the particle when t=2. 1

iv) Find the displacement of the particle when $t = log_e 3$. 2

**QUESTION 10** (12 marks) Start a new page **MARKS**

a) In the diagram, B and C are two points on the sides AD and AE of triangle ADE, such that $\angle ACB = \angle ADE = \theta$.

Also, BC = a, AC = b, AB = c, BD = 4b and CE = 2c.




i) Given that $\Delta ABC$ is similar to $\Delta AED$, deduce that $b^2 - c^2 = 2bc$. **2**

ii) By using the cosine rule in $\Delta ABC$, show that $a^2 - 2ab\cos\theta + 2bc = 0$ **2**

iii) Using the result in part (ii), show that $\cos^2\theta \geq \frac{2c}{b}$ **2**

b) Jodie won M dollars in a lottery. She invested this prize money in an account earning interest at a rate of 7.2% p.a, compounded monthly.

At the end of each month, after interest had been added, she withdrew 0.009M dollars for living expenses.

i) Let $A_n$ be the balance in her account after she withdrew her money at the end of the $n^{th}$ month. **3**
Show that $A_n = 0.5\,M\,(3 - 1.006^n)$

ii) At the end of 4 years, after making her regular withdrawals, Jodie's balance was \$1500 651. Calculate the amount she originally won in the lottery. **1**

iii) Jodie wanted her account balance to reach \$2 000 000 after a further 5 years. How much should her new withdrawals be in order to achieve this goal? **2**

END OF PAPER

**2011 Year 12, Mathematics Trial Paper Solutions**

## Question 1

a) $x^3 - 4x = x(x^2 - 4)$
$= x(x+2)(x-2)$ (1)

b)
$x^2 + y^2 - 4y = 0$
$x^2 + y^2 - 4y + 4 = 4$
$x^2 + (y-2)^2 = 2^2$ (1)

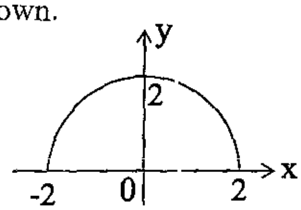
Hence, the centre of this circle is (0, 2) and its radius is 2 units. (1)

c)
$|4x+1| = 5$
Then $4x + 1 = 5$ or $4x + 1 = -5$ (1)
$4x = 4$ $\quad$ $4x = -6$
$x = 1$ or $x = -1\frac{1}{2}$ (1)

*Note*: As the absolute value equals a positive number, both solutions are valid.

d)
$\left(\sqrt{a} + \sqrt{2}\right)^2 = 5 + 2\sqrt{b}$
$a + 2 + 2\sqrt{2a} = 5 + 2\sqrt{b}$
so $a + 2 = 5$
that is $a = 3$ (1)
and $2a = b$
$\therefore b = 6$ (1)

e) $y = \sqrt{4 - x^2}$ represents a semicircle centred at the origin with a radius of 2 as shown.




Hence, the domain is $-2 \le x \le 2$ (1)

f)
$\sqrt{2}\sin\theta = 1$
hence $\sin\theta = \frac{1}{\sqrt{2}}$ (1)
so $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ in the domain $0 \le \theta \le 2\pi$ (1)

g) $\dfrac{3a - (a-4)(2a+3)}{a(2a+3)}$ (1)

$= \dfrac{3a - 2a^2 + 5a + 12}{a(2a+3)}$

$= \dfrac{8a - 2a^2 + 12}{a(2a+3)}$ (1)

## Question 2

a) i) Using the quotient rule,
let $u = \ln x$ and $v = x^2$
so $u' = \frac{1}{x}$ and $v' = 2x$
As $y = \frac{u}{v}$
then $y' = \frac{vu' - uv'}{v^2}$
$\therefore \frac{dy}{dx} = \frac{x - 2x.\ln x}{x^4}$ (1)
$= \frac{1 - 2\ln x}{x^3}$ (1)

ii) $y = (1 + \sin x)^6$
Using the chain rule
$y = u^m$
$y' = mu^{m-1} \times u'$ (1)
$\therefore \frac{dy}{dx} = 6 \times (1 + \sin x)^5 \cos x$ (1)
$= 6\cos x\left(1 + \sin x\right)^5$

b) i) $\dfrac{(2x+1)^4}{8} + c$ (1)

ii) $\int \frac{6}{3x+1}\,dx$

$= 2\int \frac{3}{3x+1}\,dx$ (1)

$= 2\ln(3x+1) + c$ (1)

iii) $\int_0^2 (px + 1)\,dx$

$= \left[\frac{px^2}{2} + x\right]_0^2$ (1)

$= 2p + 2 - (0 + 0)$

Hence, $2p + 2 = 1$

$2p = -1$

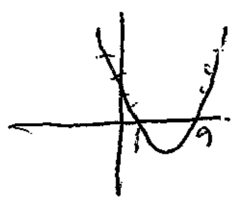
$p = -\frac{1}{2}$ (1)

c) $b^2 - 4ac > 0$

$(k+3)^2 - 16k > 0$ (1)

$k^2 - 10k + 9 > 0$

$(k-9)(k-1) > 0$ (1)




$k < 1$ or $k > 9$ (1)

QUESTION 3

a) $V = \pi \int_0^3 16 - x^2 \, dx$

$= \pi \left[16x - \frac{x^3}{3}\right]_0^3$ (1)

$= \pi \left(16(3) - \frac{(3)^3}{3}\right)$

$= 39\pi$ units$^3$ (1)

b) i) Midpoint $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\therefore E\left(\frac{5-3}{2}, \frac{1-9}{2}\right)$

$= (1, -4)$ (1)

ii) gradient BE $= \frac{6+4}{3-1} = 5$ (1)

Equation of BE is

$y + 4 = 5(x - 1)$

$y + 4 = 5x - 5$

$5x - y - 9 = 0$ (1)

iii) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$= \frac{|5 \times -1 - 1 \times 1 - 9|}{\sqrt{5^2 + (-1)^2}}$

$= \frac{|-5 - 1 - 9|}{\sqrt{26}}$

(1) $\frac{15}{\sqrt{26}} = \frac{15\sqrt{26}}{26}$ units

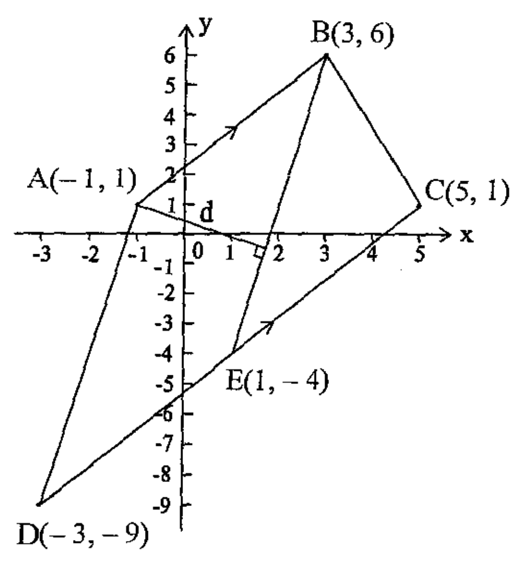
iv) Gradient of AD is $= \frac{1+9}{-1+3} = 5$

Now, as gradient of BE = gradient of AD = 5, (1)

then BE is parallel to AD. Also, AB is parallel to DE (given).

Therefore, ABED is a parallelogram

(opposite sides are parallel)




Area of the parallelogram ABED

= BE × d

$= 2\sqrt{26} \times \frac{15}{\sqrt{26}} = 30$ units$^2$. (1) (1)

c) $\frac{a}{1-r} = 18$

$\frac{\frac{9}{x}}{1-\frac{1}{x}} = 18$ (1)

$\frac{9}{x} \times \frac{x}{x-1} = 18$ (1)

$9 = 18x - 18$

$18x = 27$

$x = \frac{3}{2}$ (1)

**Question 4**

a) i) The weights that Anthony lifted each day form an arithmetic sequence with $a = 120$kg and $d = 1.5$kg.
On the $10^{th}$ day he lifted
$T_{10} = a + 9d$
$= 120 + 9 \times 1.5 = 133.5$kg (1)

ii) He lifts 150kg on a certain day that is $t_n = 150$kg that is
$150 = 120 + (n-1) \times 1.5$
$30 = (n-1) \times 1.5$
$20 = n - 1$
$21 = n$ (1)
Hence, Anthony first lifts 150kg on the $21^{st}$ day.

iii) In the first 21 days he lifted
$120 + 121.5 + \ldots + 150$
$= \frac{21}{2}(120 + 150) = 2835$kg (1)
On the remaining 7 days he lifted
$150 \times 7 = 1050$kg
So, the total weight he lifted is
$2835 + 1050 = 3885$ kg (1)
Hence, Anthony is correct as he has lifted 3.885 tonnes which is more that the weight of an elephant of 3.5 tonnes.

b)

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| f(x) | 0 | e | $2e^2$ | $3e^3$ | $4e^4$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

(1)

$I \approx \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$

$\approx \frac{1}{3}[0 + 4e^4 + 4(e + 3e^3) + 2(2e^2)]$ (1)

[illegible]

c) Interior angle sum of polygon with n sides is $(n-2) \times 180° = (20-2) \times 180°$
$= 3240°$ (1)
As all angles in a regular polygon are equal, then each angle is
$3240° \div 20 = 162°$ (1)

*Alternatively,* the sum of the exterior angles of a polygon is 360°, so each exterior angle is $360° \div 20 = 18°$.
so each interior angle $= 180° - 18° = 162°$

d) $x^2 + 2x = 8y - 9$

$(x+1)^2 = 8y - 8$

$= 8(y-1)$ (1)

(i) $V = (-1, 1)$ (1)

(ii) $a = 2$

Directrix: $y = -1$ (1)

Question 5

a) Required area =
area of rectangle – area under the curve

$= 1 \times 3 - \int_0^3 \frac{1}{2x+1}\,dx$ (1)

$= 3 - \left[\frac{1}{2}\ln(2x+1)\right]_0^3$ (1)

$= 3 - \frac{1}{2}(\ln 7 - \ln 1) = (3 - \frac{1}{2}\ln 7)\,u^2$ (1)

b) $\log_{10}(2x+4) = 1 + \log_{10} x$

$\log_{10}(\frac{2x+4}{x}) = 1$ (1)

$\frac{2x+4}{x} = 10$

$2x + 4 = 10x$

$4 = 8x$

$x = \frac{1}{2}$ (1)

c) $y = x + e^{2x}$

$\therefore \frac{dy}{dx} = 1 + 2e^{2x}$ (gradient function) (1)

When $x = 0$, $m_{tangent} = 1 + 2e^0 = 3$ (1)

Also, when $x = 0$, $y = 0 + e^0 = 1$

$\therefore$ the equation of tangent is

$y - 1 = 3(x - 0)$

$y = 3x + 1$ (1)

d) As $y = \ln x$, then $x = e^y$.

$A_1 = \int_0^a e^y dy$ (1)

$= \left[e^y\right]_0^a = e^a - e^0$

So, $e^a - 1 = 1$

$e^a = 2$

Hence, $a = \ln 2$ (1)

$A_2 = \int_a^b e^y dy$

$= \left[e^y\right]_a^b = e^b - e^a$ (1)

So, $e^b - 2 = 2$

$e^b = 4$ (1)

Hence, $b = \ln 4 = 2\ln 2$

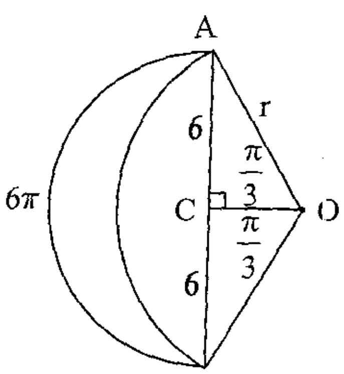
**Question 6**

a) i) As AB is the diameter of a circle with semi circumference length $6\pi$ cm then

$\frac{1}{2} \times \pi \times AB = 6\pi$

Hence, $AB = 12$cm (1)

ii)




In triangle OAC

$\sin\frac{\pi}{3} = \frac{6}{r}$, so (1)

$\frac{\sqrt{3}}{2} = \frac{6}{r}$ $\therefore r = \frac{12}{\sqrt{3}} = 4\sqrt{3}$ cm

iii) The shaded region = area of semicircle with diameter AB − area of minor segment with chord AB.

$= \frac{1}{2} \times \pi \times 6^2 - \frac{1}{2} \times (4\sqrt{3})^2 \times (\frac{2\pi}{3} - \sin\frac{2\pi}{3})$ (1)

$= 18\pi - 24 \times (\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$ (1)

$= 18\pi - 16\pi + 12\sqrt{3}$

$= (2\pi + 12\sqrt{3})\ \text{cm}^2$ (1)

b) i) $\text{LHS} = \frac{\sin^2\theta}{1 - \sin^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta}$ (1)

$= \tan^2\theta$

$= \sec^2\theta - 1$ (1)

$= \text{RHS}$

ii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{1 - \sin^2\theta} d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta$

$= \left[\tan\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ (1)

$= \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{6} - \frac{\pi}{6}\right)$

$= \sqrt{3} - \frac{\pi}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$

$= \frac{2\sqrt{3}}{3} - \frac{\pi}{6}$ (1)

c) From the graph it can be seen that $y = a\tan bx$ is undefined when $x = \pi$ that is $a\tan b\pi$ is undefined. But this occurs when $b\pi = \frac{\pi}{2}$, hence $b = \frac{1}{2}$ (1)

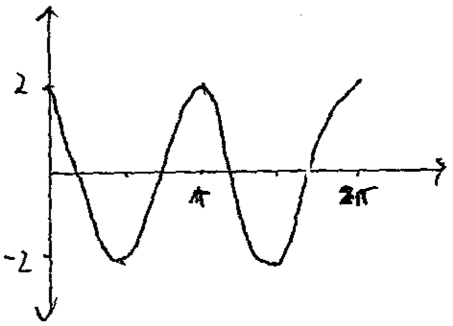
Also, from the graph we can see that when $x = \frac{\pi}{2}$, $y = 2$. By substituting this into $y = a\tan bx$ we get

$2 = a\tan\left(\frac{1}{2} \times \frac{\pi}{2}\right)$

$2 = a\tan\frac{\pi}{4}$

$\therefore a = 2$ (1)

d) i)




(1)

**Question 7**

a) Gradient of AB $= \frac{9-1}{-6+2} = \frac{8}{-4} = -2$

As the Normal is parallel to AB, then the gradient of the normal at C is $-2$ (1)

Therefore, the gradient of the tangent at C is $\frac{1}{2}$

(As tangent perpendicular to normal).

Now, $\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

$\therefore \frac{x}{2} = \frac{1}{2}$, then $x = 1$, and $y = \frac{1}{4}$ (1)

Hence, C $(1, \frac{1}{4})$.

b) In $\Delta$ CDF and $\Delta$ CBE

$\angle$ CDF = $\angle$ CBE (given)

$\angle$ DCF = $\angle$BCE

(vertically opposite angles are equal) (1)

DC = BC (equal sides of a rhombus)

$\therefore$ $\Delta$ CDF $\equiv$ $\Delta$ CBE (AAS) (1)

CF = CE (corresponding sides of congruent $\Delta$EDF and $\Delta$FBE are equal) (1)

Hence, $\Delta$ CEF is isosceles (two equal sides).

c) i) $f(x) = x^3 + 6x^2 + 15x$

$f'(x) = 3x^2 + 12x + 15$

$= 3(x^2 + 4x + 5)$

Let $f'(x) = 0$ to find the possible stationary turning points.

$\therefore 3(x^2 + 4x + 5) = 0$, that is

$x^2 + 4x + 5 = 0$

Now, $\Delta = 16 - 4 \times 1 \times 5 = -4 < 0$ (1)

As $\Delta < 0$, $f'(x)$ has no real roots and as the coefficient of $x^2$ is 3 which is positive then $f'(x)$ is positive definite.

So, the gradient is positive for all x.

Hence the curve of $y = f(x)$ is always increasing. (1)

ii) $f''(x) = 6x + 12$

Let $f''(x) = 0$ to find the possible points of inflexion.

So, $6x + 12 = 0$ $\therefore x = -2$ (1)

Test: (1)

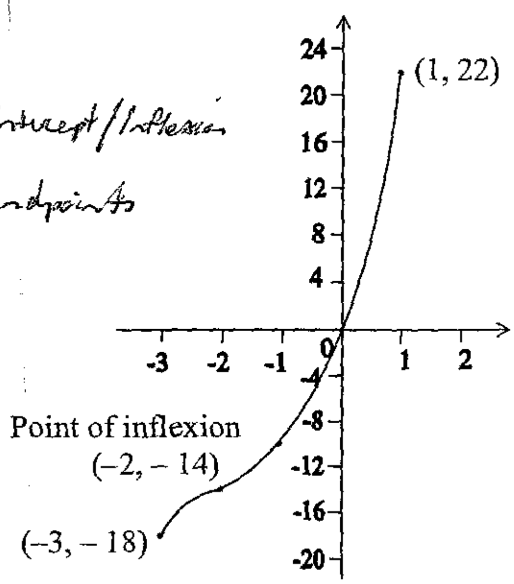
| x | −3 | −2 | −1 |
|---|---|---|---|
| f"(x) | −6 | 0 | 6 |

As the concavity changes there is a point of inflexion at $(-2, -14)$.

iii)

(1) Intercept/Inflexion

(1) Endpoints




iv) From the graph, the range is $-18 \le y \le 22$ (1)

**Question 8**

a) $f(x) = x^4 + px^3 - 6x^2 - 2$

$\therefore f'(x) = 4x^3 + 3px^2 - 12x$

$\therefore f''(x) = 12x^2 + 6px - 12$ (1)

$f(x)$ is concave up at $x = 2$

that is $f''(2) > 0$

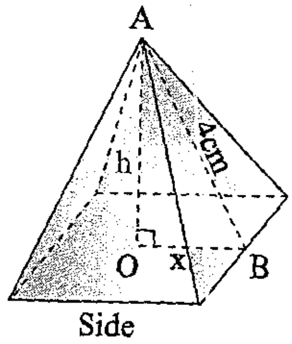
$f''(2) = 12 \times 4 + 12p - 12$

$\therefore 36 + 12p > 0$

$12p > -36$

$p > -3$ (1)

b) i)




Using Pythagoras theorem in triangle OAB, we get

$x^2 + h^2 = 4^2$

$\therefore x = \sqrt{16-h^2}$

So, the length of the side of base is

$2\sqrt{16-h^2}$ (1)

Therefore, the area of the base is

$(2\sqrt{16-h^2})^2 = 4(16-h^2)$

Hence, the volume of the prism is

$V = \frac{1}{3}\times 4(16-h^2)\times h$ (1)

$= \frac{4h}{3}(16-h^2)$

ii) $V = \frac{4}{3}(16h-h^3)$

$\frac{dV}{dh} = \frac{4}{3}(16-3h^2)$

(gradient function)

Let $\frac{dV}{dh}=0$ to find the possible stationary turning points.

$\therefore \frac{4}{3}(16-3h^2)=0$

So, $16-3h^2=0$

$h^2 = \frac{16}{3}$

$h = \pm\frac{4}{\sqrt{3}}$

But h is the height, it must be positive.

$\therefore h = \frac{4}{\sqrt{3}}$ (1)

Now, $\frac{d^2V}{dh^2} = -8h$ so when $h = \frac{4}{\sqrt{3}}$

$\frac{d^2V}{dh^2} < 0$

$\therefore$ the volume is maximum at

$h = \frac{4}{\sqrt{3}}$

So maximum volume is

$V = \frac{4}{3}\times\frac{4}{\sqrt{3}}\left(16-\frac{16}{3}\right)$

$= \frac{16}{3\sqrt{3}}\times\frac{32}{3}$

$= \frac{512}{9\sqrt{3}} = \frac{512\sqrt{3}}{27}$ units$^3$ (1)

c) (i) $P = 101e^{-kh}$

$\frac{dP}{dh} = -k\times 101e^{-kh}$
$= -kP$ } (1)

(ii) $P = 101\,e^{-kh}$

Given P = 90 when h = 1000 then

$90 = 101\,e^{-k\times1000}$

$\frac{90}{101} = e^{-1000k}$ (1)

$\ln\left(\frac{90}{101}\right) = -1000k$

$k = \ln\left(\frac{90}{101}\right) \div -1000$ (1)

$k \approx 0.0001153108....$

When $h = 5000$, $P = 101\,e^{-k\times5000}$

$P = 56.74$ kPa (1)

iii) When $P = 103$, $103 = 101\,e^{-kh}$

$\frac{103}{101} = e^{-kh}$, that is $\ln\left(\frac{103}{101}\right) = -kh$ (1)

$h = \ln\left(\frac{103}{101}\right) \div -k$

$h \approx -170.05$ (1)

Hence, the depth of the mine is approximately 170 m below sea level.

## Question 9

a) i) distance travelled = area between the velocity curve and the t- axis.

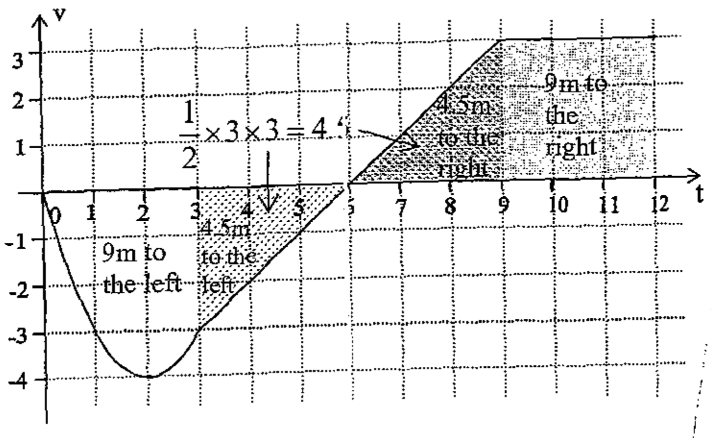
$D = \left|\int_0^3 (t^2-4t)\,dt\right|$ (1)

$= \left|\left[\frac{t^3}{3}-2t^2\right]_0^3\right|$

$= \left|\frac{27}{3}-2\times9-(0-0)\right| = 9$

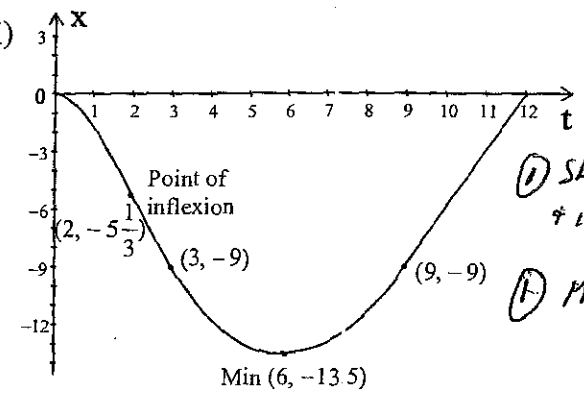
The particle travelled 9 m. (1)

ii)




The particle will return when the area under the curve and above the t- axis is equal to the area above the curve and below the t- axis. (1)
This means when the distance travelled to the left equals the distance travelled to the right.
From the graph, we could see that this occur when t = 12s. (1)

iii)




(1) Shape & intercepts
(1) Minimum

b)

i) When $t = 0$

$v = 2e^0 - 6e^0 - 1$

$= -5 \text{ m/s}$ (1)

ii) $v = 0$

$\therefore 2e^t - \frac{6}{e^t} - 1 = 0$ (1)

$2(e^t)^2 - e^t - 6 = 0$

$(2e^t + 3)(e^t - 2) = 0$

$\therefore e^t = 2$ since $e^t > 0$

$\therefore t = \log_e 2$ seconds (1)

iii) $a = 2e^t + 6e^{-t}$

when $t = 2$

$a = 2e^2 + 6e^{-2}$

$= 2e^2 + \frac{6}{e^2} \text{ m/s}^2$ (1)

iv) By integrating V with respect of time, we get $x = 2e^t + 6e^{-t} - t + c$
When $t = 0$, $x = 9$ that is
$9 = 2e^0 + 6e^0 - 0 + c$
$9 = 8 + c$
$c = 1$
$\therefore x = 2e^t + 6e^{-t} - t + 1$, (1)
when $t = \ln 3$
$x = 2e^{\ln 3} + 6e^{-\ln 3} - \ln 3 + 1$
$x = 2 \times 3 + 6 \times \frac{1}{3} - \ln 3 + 1$
$x = 9 - \ln 3$ m (1)

## Question 10

a) i) As corresponding sides in similar triangles are in the same ratio then

$\frac{AB}{AE} = \frac{AC}{AD}$ then $\frac{c}{b+2c} = \frac{b}{c+4b}$ (1)

$\therefore b^2 + 2bc = c^2 + 4bc$
$b^2 = c^2 + 2bc$ (1)
Hence, $b^2 - c^2 = 2bc$

ii) In Δ ABC,
$c^2 = a^2 + b^2 - 2ab\cos\theta$ (1)
$c^2 - b^2 = a^2 - 2ab\cos\theta$
$\therefore -2bc = a^2 - 2ab\cos\theta$ (1)
$a^2 - 2ab\cos\theta + 2bc = 0$

iii) $a^2 - 2b\cos\theta\, a + 2bc = 0$ is a quadratic equation which will only have solutions if $\Delta \geq 0$

i.e. $4b^2\cos^2\theta - 4 \times 1 \times 2bc \geq 0$ (1)

$b^2\cos^2\theta - 2bc \geq 0$

$\cos^2\theta - \frac{2c}{b} \geq 0$ (1)

$\cos^2\theta \geq \frac{2c}{b}$

b) i) 7.2 % pa = 7.2 % ÷ 12 = 0.006 per month

Balance at end of the 1st month is

$A_1 = M \times 1.006 - 0.009M$

Balance at end of the 2nd month is

$A_2 = (M\times 1.006 - 0.009M) \times 1.006 - 0.009M$

$= M \times 1.006^2 - 0.009M \times 1.006 - 0.009M$

$= M \times 1.006^2 - 0.009M\,(1 + 1.006)$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Balance at end of the nth month is

$A_n = M \times 1.006^n - 0.009M\,(1 + 1.006 + \ldots + 1.006^{n-2} + 1.006^{n-1})$

But $1 + 1.006 \ldots + 1.006^{n-1}$ is a geometric series where a=1 and r = 1.006 then

$A_n = M \times 1.006^n - 0.009M\,(\frac{1.006^n - 1}{0.006})$

$A_n = M \times 1.006^n - 1.5M\,(1.006^n - 1)$

$A_n = M \times 1.006^n - 1.5M \times 1.006^n + 1.5M$

$A_n = 1.5M - 0.5M \times 1.006^n$

$A_n = 0.5M \times (3 - 1.006^n)$

ii) Balance at end of the 4th year is

$A_{48}$ = \$ 1 500 651 then

$1\,500\,651 = 0.5M\times (3 - 1.006^{48})$

$M = 1\,500\,651 \div 0.5\times (3 - 1.006^{48})$

M = \$ 1 800 000

iii) Starting from the 5th year:

Let the new withdrawal amount be W.

The balance at end of the 1st month is

$B_1 = 1\,500\,651 \times 1.006 - W$

So, the balance at end of the 2nd month is

$B_2 = (1\,500\,651 \times 1.006 - W) \times 1.006 - W$

$= 1\,500\,651 \times 1.006^2 - W \times 1.006 - W$

$= 1\,500\,651 \times 1.006^2 - W\,(1 + 1.006)$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Hence, the balance at end of the nth month is

$B_n = 1\,500\,651 \times 1.006^n - W(\frac{1.006^n - 1}{0.006})$

Given that when n = 60, $B_n$ = \$ 2 000 000

Therefore,

$2\,000\,000 = 1\,500\,651 \times 1.006^{60} - W(\frac{1.006^{60} - 1}{0.006})$

$W(\frac{1.006^{60} - 1}{0.006}) = 1500\,651 \times 1.006^{60} - 2000000$

W = \$2065.10